

Developing Algebraic Thinking in Early Grades: Case Study of the Singapore Primary Mathematics Curriculum¹

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Abstract: This paper analyses how the Singapore primary mathematics curriculum develops algebraic thinking. Analysis of the curricular materials provides evidence to show that three approaches are adopted to develop algebraic thinking in the Singapore primary mathematics curriculum: namely, problem solving, generalising and functional. These approaches are supported by three thinking processes – analysing parts and whole, generalising and specializing, and doing and undoing. The findings show that the Singapore primary mathematics curriculum provides a wide spectrum of activities which utilise different thinking processes.

Introduction

This paper discusses the approaches used to develop algebraic thinking in the Singapore primary mathematics curriculum through an analysis of the topic outcomes and content presented in curricular materials. The approaches and the thinking processes that support the development of algebraic thinking are first identified. This is followed by a discussion of the curricular materials used for analysis and the mode of analysis applied in this study. The three thinking processes that support the three different approaches to develop algebraic thinking in the curriculum are identified. These three approaches are then discussed in turn and the discussion, using content materials as illustration, highlights how a particular thinking process, if followed, helps to develop algebraic thinking.

How the Singapore Primary Mathematics Curriculum Develops Algebraic Thinking

In *Principles and Standards for School Mathematics* (NCTM, 2000), four specific goals were delineated and the related expectations for specific grade levels for the development of algebraic thinking were identified. However, algebra is not treated so explicitly in the Singapore primary mathematics curriculum. The formal introduction of algebra at primary six (12+) is specified for the final year of primary school. At this level, the emphases are on the developing of algebraic concepts and

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algebraic manipulation skills. The following goals for teaching algebra are specified in both the curriculum and the related teacher's guide:

- to use letters to represent unknown numbers;
- to write simple algebraic expressions in one variable involving one operation;
- to write simple algebraic expressions in one variable involving more than one operation;
- to find the value of a simple algebraic expression in one variable by substitution;
- to simplify algebraic expressions in one variable involving addition and subtraction;
- to solve word problems involving algebraic expressions (CPDD, 2000).

All of these goals suggest a skills-based perspective on algebra. However, analysis of curricular materials suggests that, while algebraic thinking is not mentioned specifically, activities that contribute towards the development of algebraic thinking are proposed under other components of the mathematical framework. "Understand patterns, relations and functions" is developed under the "processes" component of the curriculum. Here, thinking skills such as "classifying, comparing, sequencing, analysing parts and whole, identifying patterns and relationships, induction, deduction and spatial visualisation" are introduced. Suggested heuristics "act it out, look for pattern(s), and guess and check" (CPDD, 2000, p. 11) are included. Because "thinking skills should be consciously integrated and reinforced through the learning of mathematical concepts and problem solving" (CPDD, 2000, p. 17), it can be inferred that activities fostering algebraic thinking are included in the curriculum albeit they are not listed overtly as such. Analysis of the curricular materials provided evidence to show that three approaches to develop algebraic thinking are adopted in the Singapore primary mathematics curriculum; these are problem solving, generalising and functional. (The book *Approaches to Algebra* edited by Bednarz, Kieran and Lee, 1996, provides a detailed discussion on this subject.) These three approaches are supported by thinking processes emphasised in the Singapore primary mathematics curriculum – analysing parts and whole, generalising and specialising, and doing-undoing respectively.

Curricular materials used for analysis

Singapore children generally complete 6 years (7+ - 12+) of primary school education. The education system is centrally directed so all schools follow a common curriculum. Prior to 2001, all primary schools used the teacher's guide, as well as textbooks and workbooks written by a team from the Ministry of Education. In 2001, in line with the Ministry of Education's vision of the "Thinking Schools, Learning Nation", the content of the primary mathematics syllabus was reduced by

thirty percent in order to enable more time for the infusion of three initiatives into the mathematics curriculum - Thinking Skills, Information Technology and National Education (CPDD, 2000). From 2001 onwards, to allow for more diversity of curricular materials, schools have been at liberty to adopt textbooks commercially produced by different publishers. The books produced by the Ministry of Education will be phased out by the year 2006. The adoption of commercially produced books is on a year by year basis; that is, primary one in 2001, primary two in 2002 until primary six in 2006. All commercially produced books adhere strictly to the curriculum set by the Ministry of Education and are only available subject to approval. Approval is obtained once Ministry of Education appointed reviewers confirm that the books comply with the 'outcomes' of each topic which are clearly specified in the syllabus.

Because the textbooks produced by the Ministry of Education are currently still in use and also for the sake of consistency, the data for this study were collected from the syllabus, teachers' guides, textbooks and workbooks produced by the Ministry of Education. There are two textbooks, four workbooks, and two accompanying teacher's guides for each of primary levels one to four. In primary levels five and six, the number of workbooks is reduced to two. The teacher's guide, textbooks and workbooks are referenced with the letters A or B and the accompanying level. For example, TG3A and TB3A refer to teacher's guide and textbook respectively for primary level 3 while WB3A Part1 designates part one of workbook 3A.

Each teacher's guide is divided different sections. Teaching notes for the teachers are also supplemented by additional resources which include activities that develop mathematical thinking, encourage mathematical investigation and problem solving and are labeled as 'Mathematical Thinking', 'Mathematical Investigations' and 'Problem Solving' respectively. Topics and concepts are introduced and developed through sections called units. Mathematical Thinking, Mathematical Investigations and Problem Solving Tasks that require pupils to apply concepts and skills taught within and across units are presented in each unit of the teacher's guide. Teachers can choose to use such tasks to consolidate skills taught in that unit or to revise concepts and skills taught in previous units.

Mode of analysis

The primary mathematics syllabus is divided into two sections – topic outcomes and teaching notes. Under topic outcomes, the objectives of each topic are clearly stated and the teaching notes provide examples of activities that are to be included and also those to be excluded. The topic outcomes and activities were analysed to identify which particular approach these activities fall under. Once this was completed the teacher's guide, textbooks and workbooks at each level were

examined to identify how the approach was developed and to what extent this approach was adopted.

In summary, the curricular materials were analysed to identify the thinking processes emphasised, the approach adopted to develop algebraic thinking, the objectives of recommended activities, and the extent of the adoption.

Processes

Analysis of the curriculum suggests that three thinking processes – analysing parts and whole, generalising and specializing, and doing and undoing - are emphasised in the Singapore primary mathematics curriculum. Because the model method which utilises part-whole reasoning is an integral part of the curriculum, this section thus begins with an introduction to the model method. This is then followed by the discussion of the three thinking processes.

The model method

In the Singapore mathematics curriculum, the model is a structure that encapsulates all existing procedural relationships present in a given problem. This model method must not be confused with the modelling approach discussed by Nemirovsky (1996b) and Heid (1996b). Rather the model is a structure comprised of rectangles and numerical values that represent all the information and relationships presented in a given problem. The rectangles replace the unknown represented by letters in equations. The rectangle, known as a unit, becomes the “generator” (Bednarz & Janvier, 1996) of the model about which other relations in a given problem are constructed. There are two stages to learning how to construct a model for a problem. At the first stage of learning, the part-whole concept is developed and this is taught at primary one and two (See Figure 1, from TG2A, 1995, p. 23- 24). In the second stage that begins around primary three, models based on the concept of proportional reasoning (Figure 2, from WB4A Part 2, 2000, p. 22) are constructed.

Analysing parts and whole

Recognising and articulating the parts that together form a whole is a central theme in the mathematics curriculum. This process emphasises the identification of the part or unknown that is the ‘generator’ (Bednarz & Janvier, 1996) of relationships in a given problem. Identifying the generator, and abstracting and articulating the procedural relationships between the parts and the generator are crucial to problem solving via the model method.

Generalising and specialising

In specialising, previously acquired skills are used to explore and get a sense of the structure or idea supporting a particular mathematical concept or physical object.

Generalising takes place when one is able to identify the pattern that is consistent among several instances of the supporting idea. Within generalising and specialising, two processes fundamental to mathematical thinking, are such sub-processes identified by Mason: “detecting sameness and difference, making distinctions, repeating and ordering, and classifying and labelling” (1996, p. 83).

Doing and undoing

Being able to undo a process that produced a particular goal helps provide greater insight into the nature of the operation that produced that goal (Open University, 1982; Driscoll, 1999). Such reversibility in process is a necessary part of effective algebraic thinking. There is a widespread presence of the theme of doing and undoing in the Singapore primary mathematics curriculum.

These processes of analysing parts and whole, generalising and specializing, and doing – undoing correspond directly with the three approaches of problem solving, generalising and functional, used to develop algebraic thinking in the Singapore primary mathematics syllabus. I address these next.

Problem-solving approach to developing algebraic thinking

In the problem-solving approach to algebraic thinking, the equation is the problem-solving tool and constructing appropriate equations is regarded as of paramount importance. (See Bednarz & Janvier, 1996 for a full discussion on this approach.) The construction of equations using letters as unknowns to solve a given problem is introduced at secondary one (13+). However, primary pupils are taught to solve similar algebraic type problems by drawing models that represent the problem situation.

Model method as a tool for problem solving

Figure 1 suggests how teachers could develop the part-whole concept and how the model method is introduced at primary two. In this example, pictures of cars are initially used to model the problem situation and the cars are then replaced by the more abstract rectangles; in this case one rectangle represents one car, but this need not be the case. This progressively more complex approach (minus the doing-undoing aspect) begins in the primary one teacher’s guide. This example shows how the model method is used to solve an arithmetic problem where pupils worked with known values to solve the unknown (the total number of cars). As pupils progress through the primary years, the model method is used to solve algebraic problems involving unknowns, part-whole concept and proportional reasoning (Figure 2). In each case the rectangles allow pupils to treat the unknowns as if they are knowns and here each rectangle is a unit that represents more than one object.

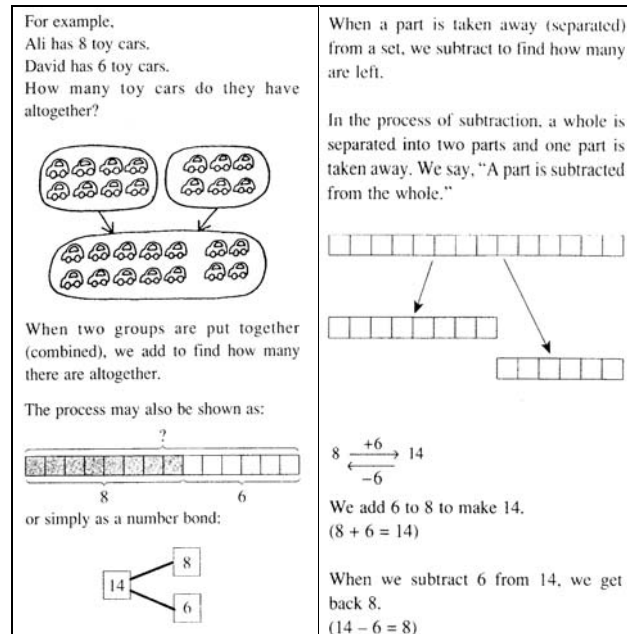


Figure 1. Model method: part-whole.

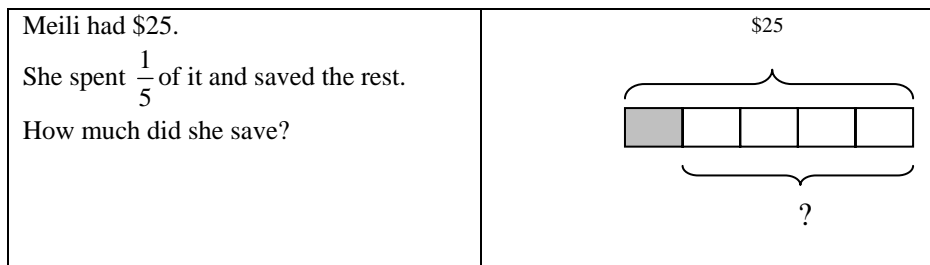


Figure 2. Model method: proportional reasoning.

Problems are made more challenging when proportional reasoning and relational quantities are involved as can be seen in the example in Figure 3 from the primary five textbook (TB5A, 1999, p. 23). The term unit is used to represent an unknown and in this case Samy's rectangle or unit is the generator of all existing relationships presented in the problem – Raju's rectangle is dependent upon Samy's with Raju's share represented by a unit identical to Samy's plus another rectangle representing the relational portion of \$100 more. Using the model drawing, a pictorial equation

representing the problem is formed and if the letter x replaces Sammy's unit, then the algebraic equation $x + x + \$100 = \410 is produced. The use of the rectangle as a unit representing the unknown provides a pictorial link to the more abstract idea of letters representing unknowns. The entire structure of the model can be described as a pictorial equation.

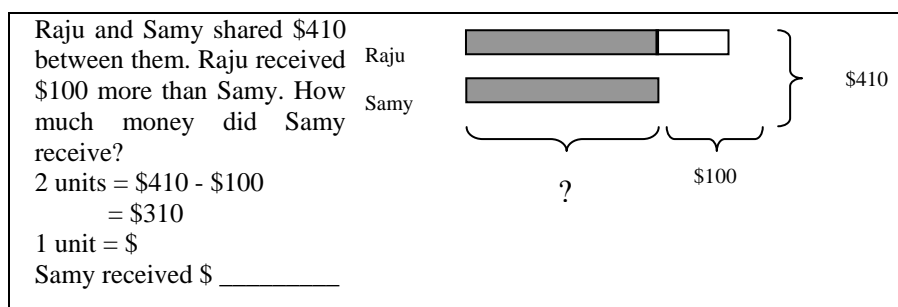


Figure 3. Complex proportional reasoning.

As transformation is only taught in secondary one, primary pupils are shown procedures to solve for the unknown set up in a pictorial equation. First, rather than forming an equation using “forward operations” (Kieran, Boileau, & Garançon, 1996, p. 264) ($2 \text{ units} + 100 = 410$) - the equation where the relational aspect of adding 100 is undone is constructed - $2 \text{ units} = 410 - 100$. Second, the value of one unit is evaluated by carrying out the doing and undoing process; that is, if 2 units is equivalent to 310, what do you do to find the value of one unit?

The model method, as a pictorial tool, can be used to solve increasingly complex problems. The following are items from the Primary School Leaving Examination mathematics paper (12+). As formal algebra has not been taught, the model method used to construct appropriate pictorial equations becomes the tool to solve these problems.

3 children sold a number of funfair tickets.

Ailin sold $\frac{1}{8}$ of these tickets. Brian sold 4 tickets more than Ailin.

Chandra sold 32 tickets. How many tickets did they sell altogether?
(MOE, 2002, p. 16)

There are 48 more fruit tarts than egg tarts. After $\frac{5}{6}$ of the fruit tarts and $\frac{3}{4}$ of the egg tarts were eaten, there were 33 tarts left. How many egg tarts were there at the beginning and what fraction of the eaten tarts were fruit tarts? (2002)

Summary: Model method

The problem solving approach pervades the entire primary mathematics curriculum. Through the model method, pupils with no knowledge of formal algebra are provided with a tool to construct pictorial equations to solve increasingly challenging word problems, involving simple part-whole relationships as far as those that require proportional reasoning.

Generalisation approach

This approach advocated by Mason (1996) does not suggest that pupils are only required to generalise a pattern by algebraic means. Rather the generalisation proposed by Mason underpins all mathematical learning of pupils. For example, pupils who use concrete materials to learn addition and subtraction of three-digit numbers generalise the algorithm they have constructed to addition and subtraction of four-digit numbers, now without having to depend on concrete materials. Generalising and specialising are the processes supporting this approach.

Activities requiring pupils to identify, understand and extend repeating geometrical patterns as well as extend growing numerical patterns pervade the entire primary mathematics curriculum. Increasingly more challenging numerical and geometric patterns are presented at each subsequent primary level. Pattern identification activities presented in the textbooks are supplemented by more challenging ones in the teacher's guide. The discussion in this section is divided into two subsections – recognition of number patterns and recognition of geometric patterns to demonstrate how the primary mathematics syllabus supports each set of activities.

Recognition of number patterns

Items that require pupils to continue growing number patterns are not presented under the specific goal of continuing growing number patterns. Instead such items are an inherent part of the teaching of core topics such as 'Number Notation and Place Values' and 'Comparing and Ordering of Numbers'. Figure 4 shows some of the specific outcomes for primary one (CPDD, 2000, p. 33)

Under the topic 'Whole Numbers', the tasks listed in Figure 5 require primary one pupils (7+) to identify whether the number sequences are of an increasing or decreasing order and to understand that the pattern is preserved either by adding or subtracting one from the preceding number. Once pupils can manipulate such number sequences and have developed a sense of these simple number patterns, they are offered number sequences of increasing complexity at primary two. At this level, the thinking processes of detecting sameness and difference, making distinctions, repeating and ordering are repeated (Figure 6).

Level: Primary 1	Whole Numbers
Topics/outcomes	Teaching notes
Pupils should be able to a) count to 100	a) Include activities such as completing sequences of consecutive numbers Examples: 15, 16, 17 ____, ____ 24, 23, 22, ____. ____ Include counting in tens and completing sequence Examples: 10, 20, 30, ____, ____
c) recognise the place value of numbers	
3 Comparing and Ordering h) compare numbers up to 100 arrange numbers in increasing and decreasing order	

Figure 4. Selected primary 1 outcomes.

<p>What comes next? (TB 1A, p. 14) 6, 7, 8, 9 What are the missing numbers? (TB1A, p. 15) (a) 4, 5, 6, ____, ____, 9, ____ (b) 9, 8, 7, ____, 5, ____, ____ .</p>	
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Figure 5. Primary one – number pattern activity

Level: Primary 2	Whole Numbers
Topics/outcomes	Teaching notes
Pupils should be able to a) count to 1000 c) recognise the place value of numbers (hundreds, tens, ones) d) compare and order numbers up to 1000	c) Include activities to consolidate concept of place value such as completing number sequences Examples: 103, 104, 105, ____, ____ 32, 42, 52, ____. ____ 232, 222, 212, ____, ____ 440, 340, 240, ____, ____

Figure 6. Selected primary 2 outcomes.

Here number sequences where the digits in either the tens place or hundreds place vary are introduced. Such number sequences help raise and consolidate pupils' awareness of place value that satisfies the goal of the topic. The following two tasks are set in the primary two (Figure 7, from TB2A, 1995, p. 101) and primary five (Figure 8, from TB5A, 1999, p. 10) textbooks where the expectations are for pupils to learn numbers up to 1000 and 10 million respectively. Also, to encourage pupils to utilise their knowledge of multiplication number bonds, growing numbers sequences that require skip counting are introduced at primary two. Pupils can either recall the multiplication number bonds of three or they can skip count (e.g., 3, 6, 9, 12 ...). How they choose to complete the number sequence is immaterial as invariably they need to recognise a pattern.

103, 104, 105, _____, _____ 32, 42, 52, _____, _____ 232, 222, 212, _____, _____ 440, 340, 240, _____, _____ (CPDD, 2000, p. 43)	In the review exercise, What are the missing numbers? (a) 3, 6, 9, 12, _____, _____, _____, 24, _____, 30 (b) 500, 490, 480, _____, _____, _____, 440, _____, 420
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Figure 7. Primary two - number pattern activity

How are the processes specialising and generalising applied? The intent of providing such number sequences (e.g. 42 668, 43 668, _____, _____, 46 668) is that, while pupils hold and juxtapose these numbers in their head, they see that numbers can still change in a particular manner even though certain digits remain the same.

Complete the following number patterns. (a) 42 668, 43 668, _____, _____, 46 668 (b) 70 500, 71 500, 72 500, _____, _____ (c) 83 002, 93 002, _____, _____, 123 002 (d) 5 632 000, 5 642 000, _____, _____, 5 672 000 (e) 9 742 000, 8 742 000, _____, 6 742 000, _____
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Figure 8. Primary five - number pattern activity

Pupils use their previously acquired concepts of place value and addition of thousands to explore the underlying structure of the given sequence. Pupils specialise by manipulating the first two numbers and once they have a sense of the pattern (Mason, 1996; Open University, 1982), they then construct a rule that enables them to extend the number sequence. Pupils' extension of the number sequences is their attempt at generalisation. Their rule is verified when it generates the next given number of 46 668 or by retesting this rule, specialising the rule on numbers at the beginning of the sequence.

Number pattern activities do not focus solely on whole numbers. Numerical patterns using rational numbers are also included for pupils to explore. Pupils first encounter number pattern activity with rational numbers at primary four. At this level, pupils are introduced to the concept of decimal fractions up to the thousandth place-value. Some of the number patterns involving decimal fractions are shown in Figure 9 (WB4B, Part 1, p. 20). Pupils are required to complete the decimal fraction sequences where the digit in the tenth place increases or decreases by one-tenth or five-tenths. The degree of difficulty of such number pattern activities increases when pupils are required to understand and complete decimal fraction sequences where the digit in the hundredths place either increases or decreases by one-hundredth or five-hundredths. Again pupils are involved in the specialising and generalising processes.

4. Complete the following number patterns.
- (a). 0.8, 0.9, _____, 1.1, _____, 1.3
 - (b). 1, 1.5, 2, 2.5, _____, _____, 4
 - (c). 3, 2.9, 2.8, _____, 2.6, _____, 2.4
 - (d). 10, 9.5, 9, _____, 8, _____, 7
 - (e). 0.05, 0.1, 0.15, _____, 0.25, _____, 0.35
 - (f). 0.45, 0.4, 0.35, _____, _____, 0.2
 - (g). 0.02, 0.04, 0.06, _____, 0.1, _____, 0.14
 - (h). 10, 9.95, 9.9, _____, 9.8, _____, 9.7

Figure 9. Number patterns involving rational numbers

Additional resources that are more challenging than those offered in textbooks are provided in the teacher's guide. The task "Mathematical Investigations 8" (Figure 10) provided in the primary two teacher's guide (TG2A, 1995, p. 54) requires pupils to relate elements in a pattern to their positions in the pattern and generalise those relationships. To solve this problem, pupils first have to note that while the numbers are increasing, the digits in the ones place, however, change in a fixed pattern and each child receives cards with certain digits in the ones place. Hence pupils note that David will always have number cards ending either in 3 or 8. The pupils then generalise that this sameness in the ones digit will mean that David will get the number card 453. Pupils can test the rule on specific examples by asking which child will get the number cards 13 or 28 – here pupils specialise on specific examples.

Once new concepts are taught, activities where the goal is to apply the newly taught concepts to understand patterns are provided. For example, after learning about factors and multiples, besides being able to list the factors and multiples of numbers, opportunities are provided for primary four pupils to apply their

knowledge of factors and multiples to look for relationships that exist between numbers. The activity in Figure 11 (TB4A, 2000, p. 25) guide pupils to generalise about numbers that are factors of 2 and 5. Question 8 of the same activity encourages pupils to look for relationships between numbers that are multiples of three and the sum of the digits of such numbers. A similar but extended activity is provided in *Navigating through Algebra in Grades 3 – 5* (Cuevas & Yeats, 2001, p. 9).

Mary	John	David	Tom	Sally
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
453	570	602	746	829

The number cards 1 to 1000 are distributed to five children in this order

Who will get these cards?

Figure 10. Mathematical Investigation 8

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

(a) When a number is a multiple of 2, its ones digit is 0, 2, or .

(b) When a number is a multiple of 5, its ones digit is or .

8. Write any number that is a multiple of 3.
Find the sum of the digits of the number.
Is the sum a multiple of 3?

25

Figure 11. Number patterns involving factors.

Geometric patterns

Primary one pupils are taught to identify four shapes - triangle, square, rectangle and circle - by sight. They are provided with activities where they are expected to sort shapes according to three attributes – size, colour and shape (TB1A, 1994, pp. 72 – 74). Once they can recognise the shapes, they are expected to complete geometric patterns.

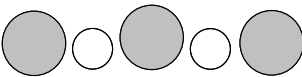

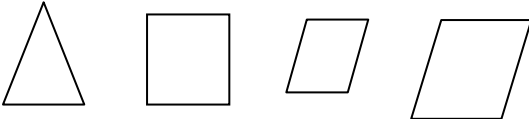
Level: Primary 1	Geometry
Topics/outcomes	Teaching notes
2. Patterns c) complete patterns according to Shape Size Colour Two of the above attributes	c) Include use of figures/cut-outs to illustrate the pattern Example:  (i) What is the colour of the fifth circle? (ii) Draw the 6 th shape. (CPDD, 2000, p. 39)
Complete patterns with 3-D objects Cube Rectangular block Cone Cylinder	d) Use concrete objects Exclude use of the words ‘cube’, ‘cone’, ‘cylinder’ in written or verbal form.
Level: Primary 2	Geometry
complete patterns according to Shape Size Orientation Two of the above attributes	Include activities for pupils to identify the patterns and relationships Example:  Draw the next shape. (CPDD, 2000, p. 50). Include activities for pupils to create patterns using cut-outs or IT.

Figure 12. Selected primary 1 and 2 geometry outcomes.

The list of outcomes for the geometry topic for both primary one and two (Figure 12) shows that completing geometric patterns is a clearly stated goal. Primary one and two pupils are expected to complete repeating patterns with two-dimensional objects and to address the question ‘What comes next in the pattern?’ Completing

patterns constructed using 3-dimensional objects is restricted to working with actual solids. No pattern generation activities using pictorial or written tasks are provided because pupils may have difficulty drawing or naming the solids.

The syllabus suggests how algebraic thinking activities involving shapes can be continued in upper primary as for the example shown in Figure 13 (CPDD, 2000, p. 134) where the aim of the activity is to compare and classify shapes according to their properties and also to apply the heuristic ‘look for patterns’.

Classify and compare the following set of figures in as many ways as possible, stating the reason(s) for the classification each time.	
	
Equilateral triangle Square Rhombus Parallelogram	
Thinking Skill: Comparing and Classifying Problem Solving Heuristics: Look for Patterns	
Classification	Reason
e.g. 1 a) Equilateral triangle and Square b) Rhombus and Parallelogram 2.	All angles in each figure are equal. Not all angles in each figure are equal.
<i>Figure 13. Algebraic thinking extended to geometric shapes.</i>	

Summary: Generalisation approach

Pupils are provided with both numeric and geometric pattern recognition activities. Such activities require pupils to specialise and then generalise the rule they construct to continue the pattern they see. The propensity of such activities in the curricular materials suggests that the Singapore primary mathematics syllabus places great importance on the thinking processes – generalising and specialising.

Functional approach

The functional approach to algebra does not imply the study of functions per se. Instead, the research literature (Heid, 1996a; Kieran, Boileau, & Garançon, 1996; Nemirovsky, 1996a) suggests that the functional approach emphasises the use of

letters as variables rather than as unknowns. In addition, it draws attention to how a change in the input affects the behaviour of the value of the function or the output. Furthermore an operational, process-oriented way is used to express the functional relations. When using a functional approach to solve a given algebraic problem, a rule is constructed to evaluate the relationship between the pairs of input-output variables. To formulate a rule for a given relationship involves a certain amount of generalisation. Hence there is also some intersection between the functional and generalisation approach (Kieran et. al, 1996).

While research literature (e.g., Kieran et al., 1996) underscored the importance, for pupils in formal algebra classes, of constructing solutions using “forward operations”, researchers (Mason,1988; Driscoll, 1999) considered it vital for pupils in pre-algebra classes to develop the process ‘doing - undoing’ as this becomes a bridge to more formal algebraic thinking. Mason (1988) explained that engaging pupils in the doing -undoing process will often lead them to useful mathematical ideas about the properties of operations. Pupils can solve this problem ‘*When 5 is added to 4 times the number in my head, I get 33. What is the number?*’ in two ways. They can solve for the unknown number either by constructing an equation or by undoing the operations. Here pupils understand the relationships between addition and subtraction, multiplication and division. For the solution to be considered algebraic, Usiskin (1988) stressed that a rule using “forward operations” should be constructed, rather than one with using inverse operations. The solution to the problem is algebraic when the equation $4 \times \text{the number} + 5 = 33$ (forward operations) is constructed. The solution is arithmetical when the inverse operations of subtracting 5 from 33 and dividing the result by 4 are used, that is, when the pupil undoes the operations. However other research (Kieran, 1983) showed that pupils who ‘undo’ the operations were not better at constructing equivalent equations while in the process of solving for the unknown. Nevertheless the Singapore primary mathematics curriculum offers the choice to develop these ideas: doing-undoing and construction of the rule using “forward operations”.

Notes from the teacher’s guide show that the process of ‘doing-undoing’ is emphasised in the Singapore teacher’s guide, particularly when teachers first introduce the four operations. For example, in the unit ‘Addition and Subtraction’, notes in the primary two teacher’s guide suggest that teachers highlight the relationship between addition and subtraction as evidenced by the example in Figure 1 (TG2A, 1995, p. 23 – 24). The relationships between the operations addition and subtraction, multiplication and division are highlighted by the arrow diagram in Figure 14.

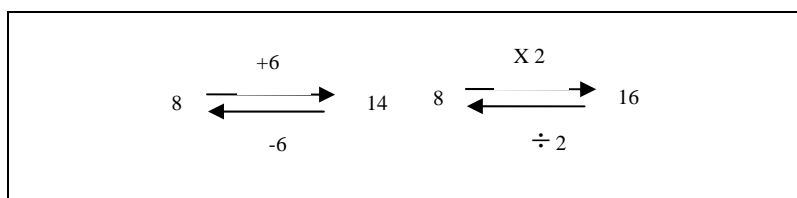


Figure 14. Arrow diagram to highlight doing-undoing relationship.

The primary mathematics curriculum provides pupils with activities in which they develop the pointwise notion of function where they look for the relation between binary points – the input and output of a problem situation. This notion is developed informally through supplementary activities provided in the primary two teacher's guide as shown in Figure 15 (Mathematical Thinking 73, TG2A, p. 18; 1995) and also as part of the specific goals of learning and committing the multiplication bonds to memory and to solve word problems (Figure 16).

Study the pattern and find the missing number.		
1	→	99
2	→	199
3	→	299
4	→	399
5	→	<input type="text"/>

Figure 15. Mathematical Thinking 73 – a supplemental activity

The functional approach provides primary three pupils with experiences where they not only extend number patterns written sequentially, but also with tasks that require them to look for patterns where the data are presented in tables. Here they have to understand the pattern and use the pattern to devise the rule linking numbers in one row/column with numbers in another row/column. Such activities provide pupils with an informal introduction to pointwise functions as they have to identify the rule that transforms one set of numbers into another set. In the Singapore mathematics curriculum, pupils are guided into such an activity through a context. The following task is presented to pupils as an introduction to the multiplication table of 7 (Figure 16, TB3A, 1999, p. 76).

Mr Samy made this table to help him collect						(a) Aihua bought 2 cakes. How much did she pay?
Number of cakes	1	2	3	4	5	(b) Mrs Li ordered 4 cakes for a party. How much did she pay?
cost	7	14	21	38	35	(c) Siti paid Mr Samy \$35. How many cakes did Mr Samy give her?
money.						(d) How many cakes could Mr Fu buy with \$42?

Figure 16. Notion of function specific to curriculum outcomes.

Questions (c) and (d) are different from questions (a) and (b) as both (c) and (d) require pupils to undo the rule of multiplying by 7.

The complexity of the doing and undoing process is increased at Primary 3. Mathematical Thinking 124 task in the teacher's guide (TG3A, 1999, p. 64) challenges pupils to determine the number after two operations.

*Add 30 to a number. Then subtract 35 from the sum. The answer is 27.
What is the original number?*

At primary six, the informal introduction of functions adopts a developmental approach – pictorial to symbolic - to introduce the non-recursive rule and the concept of letter as variable. Research literature suggests (Kieran, 1997; Kieran et al., 1996) that it is much easier to complete the table in Figure 17 using the recursive rule of $T_{n+1} = T_n + 4$ than the non-recursive form $T_n = n \times 4$. The example in Figure 17 provided in the primary six textbook (TB6A, 2000, p. 8), requires the construction of a forward non-recursive rule that, when applied onto an input, produces the output. The cartoon provides both the rule using forward operation, $4 \times n$, and its equivalent structure, $4n$.


The syllabus also provides a specific example which can be used to develop pattern recognition and to find a rule that will determine the input given the output (CPDD, 2000, Example 9, p. 135; Figure 18).

Summary: Functional approach

Through the functional approach, the Singapore primary mathematics curriculum provides pupils with opportunities to deepen their knowledge of the four mathematical operations, to learn and apply their knowledge of specific operations and their related number bonds, and ultimately to apply the thinking skill of 'identifying patterns and relationships'. The content coverage of this approach is

extensive as it starts from primary two and culminates in the construction of the forward, non-recursive rule and the introduction of letters as variables in formal algebra at primary six.

4. There are 4 apples in each packet.



(a) How many apples are there in n packets?

Number of packets	Total number of apples
1	$4 \times 1 = 4$
2	$4 \times 2 = 8$
3	$4 \times 3 = 12$
4	$4 \times 4 = 16$
5	$4 \times 5 = 20$
n	$4n$

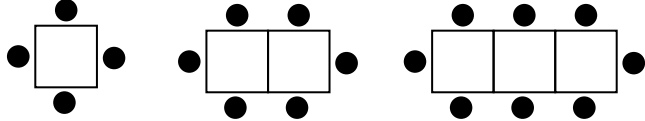
We write $4 \times n$ as $4n$.

(b) If $n = 8$, how many apples are there altogether?
 (c) If $n = 11$, how many apples are there altogether?

Figure 17. Use of formal non-recursive rule.

A square table can seat 4 people. How many such square tables, arranged to form a long table, are needed to seat 30 people?

Method 1
 Thinking Skill: Identifying pattern and relationships, Induction
 Problem Solving Heuristics: Use a Diagram, Make a Systematic List and Look for a Pattern



No. of tables	1	2	3	4	5	6	...	14
No. of people		4	6	8	10	14	...	30

14 tables are needed.

Figure 18. Example 9 – rule finding.

Conclusion

The Singapore primary mathematics curriculum, by providing a wide spectrum of activities which utilise different thinking processes, engages pupils in algebraic thinking. The goals of the activities include problem solving, identifying patterns, constructing rules and functions for number sequences and using letters as variables. The content coverage is comprehensive. The spiral nature of the curriculum means

that content covered at a lower level is revisited, each visit ensuring that pupils engage with similar ideas and concepts, each at a higher and more complex level. Through the three approaches – problem solving, generalisation and functional - the Singapore primary mathematics curriculum provides comprehensive and detailed support for algebraic thinking within the daily mathematics curricular activities.

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